Observability and Estimation of Distributed Space Systems via Local Information-exchange Networks

Amirreza Rahmani, Mehran Mesbahi[‡]

Department of Aeronautics and Astronautics

University of Washington

and

Nanaz Fathpour, Fred Y. Hadaegh[*]

Jet Propulsion Laboratory

California Institute of Technology

Abstract

In this work, we develop an approach to formation estimation by explicitly characterizing formation's system-theoretic attributes in terms of the underlying inter-spacecraft information-exchange network. In particular, we approach the formation observer/estimator design by relaxing the accessibility to the global state information by a centralized observer/estimator- and in turn- providing an analysis and synthesis framework for formation observers/estimators that rely on local measurements. The novelty of our approach hinges upon the explicit examination of the underlying distributed spacecraft network in the realm of guidance, navigation, and control algorithmic analysis and design. The overarching goal of our general research program, some of whose results are reported in this paper, is the development of distributed spacecraft estimation algorithms that are scalable, modular, and robust to variations in the topology and link characteristics of the formation information exchange network. In this work, we consider the observability of a spacecraft formation from a single observation node and utilize the agreement protocol as a mechanism for observing formation states from local measurements. Specifically, we show how the symmetry structure of the network, characterized in terms of its automorphism group, directly relates to the observability of the corresponding multi-agent system. The ramification of this

[‡] Email: {arahmani,mesbahi}@aa.washington.edu.

[*] Email: {nanaz.fathpour,fred.y.hadaegh}@jpl.nasa.gov.

notion of observability over networks is then explored in the context of distributed formation estimation.

Index Terms

Distributed space systems, networked systems, observability, automorphism group, agreement dynamics, algebraic graph theory.

I. Introduction

Distributed space systems rely on a signal transmission network among multiple spacecraft for their operation. The network consists of relative sensing and inter-spacecraft communication, which will be collectively referred to as the formation information exchange network. Control and coordination among multiple spacecraft in the formation is facilitated via this network. The dependency of the formation guidance, navigation, and control architecture on the underlying formation network will be more pronounced as these formations become complex and their science objectives dictate higher levels of precisions and dynamic reconfigurations. We will develop a unified approach to formation estimation by explicitly characterizing formations system theoretic attributes in terms of the underlying inter-spacecraft information-exchange network. In particular, we approach the formation observer/estimator design by relaxing the accessibility to the global state/control information- and in turn- provide an analysis and synthesis framework for formation observers/estimators that rely on local and/or incomplete measurements. The novelty of our approach hinges upon the explicit examination of the underlying distributed spacecraft network in the realm of guidance, navigation, and control algorithmic analysis and design. The main goal of this research is the development of distributed spacecraft estimation algorithms that are scalable, modular, and robust to variations in the topology and link characteristics of the formation information exchange network

A distributed spacecraft is a prime example of a networked dynamic system. A networked system is a collection of dynamic units that interact over an information exchange network for its operation. Such systems are ubiquitous in diverse areas of science and engineering. Examples other than distributed space systems include physiological systems and gene networks [12], large scale energy systems, and multiple space, air, and land vehicles [1], [2], [27], [38], [37]. There is an active research effort underway in the control and dynamical systems community to study these systems and lay out a foundation for their analysis and synthesis [6], [7], [9]. As a result, over the past few years, a distinct area of research at the intersection of systems theory and graph theory has emerged. An important class of problems that lies at this intersection pertains to the *agreement* or the *consensus problem* [4], [15], [28], [30], [39]. The agreement problem concerns the development of processes by which a group of dynamic units, through local interactions, reach a common value of interest. As such, the agreement protocol is essentially an unforced dynamical system whose trajectory is governed by the interconnection geometry and the initial condition for each unit.

Our goal in this paper is to utilize the agreement protocol as a mechanism for observing formation states from *local measurements*. This accomplished by introducing nodes in the agreement protocol that serve as observation posts for the dynamics. The network observability from local measurements has not generally been considered in the literature- exceptions include the work of Olfati-Saber and Shamma in the context of consensus filters [31]. However, the dual of the observability problem, namely controllability of leader-follower multi-agent systems, has recently been considered in by Tanner [36], Ji *et al.* [18], and Rahmani and Mesbahi [34]. In the present work, we further explore the ramifications of this graph-theoretic outlook on multi-agent systems observability. Specifically, we examine the role of the graph Laplacian eigenvectors and the graph automorphism group for the observabilty of networks augmented with a single observation post.

The paper begins with the general form of the agreement dynamics over networks. Next, we introduce transformations that, given the location of the observation node, produce the corresponding observed linear time-invariant system. The study of the observability for single-observer systems is then pursued via tools from algebraic graph theory. In this avenue, we provide a sufficient graphical condition in terms of graph automorphisms for the system's unobservability. The ramification of the above network-theoretic outlook toward observability is the explored in the realm of formation estimation algorithms.

II. Notation and Preliminaries

In this section we recall some basic notions from graph theory, followed by the general setup of the agreement problem for multi-agent networks.

A. Graphs and Their Algebraic Representation

Graphs are broadly adopted in the multi-agent literature to encode interactions in networked systems. An undirected graph $\mathcal{G}$ is defined by a set $\mathcal{V}_\mathcal{G} = \{1, \dots n\}$ of *nodes* and a set $\mathcal{E}_\mathcal{G} \subset \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$ of *edges*. Two nodes i and j are *neighbors* if $(i, j) \in \mathcal{E}_\mathcal{G}$; the neighboring relation is indicated with $i \sim j$, while $\mathcal{P}(i) = \{j \in \mathcal{V}_\mathcal{G} : j \sim i\}$ collects all neighbors of node i. The *degree* of a node is given by the number of its neighbors; we say that a graph is *regular* if all nodes have the same degree. A *path* $i_0 i_1 \dots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \dots, L$, and a graph $\mathcal{G}$ is *connected* if there is a path between any pair of distinct nodes. A subgraph $\mathcal{G}'$ is said to be *induced* from the original graph $\mathcal{G}$ if it can be obtained by deleting a subset of nodes and edges connecting to those nodes from $\mathcal{G}$.

The *adjacency* matrix of the graph $\mathcal{G}$, $A(\mathcal{G}) \in \mathbb{R}^{n\times n}$, with n denoting the number of nodes in the network, is defined by

$$[\,\mathcal{A}(\mathcal{G})\,]_{ij} := \begin{cases} 1 & \text{if} \quad (i,j) \in \mathcal{E}_\mathcal{G} \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{G}$ has m edges and is given an arbitrarily orientation, its node-edge *incidence* matrix $\mathcal{B}(\mathcal{G}) \in \mathbb{R}^{n\times m}$ is defined as

$$[\,\mathcal{B}(\mathcal{G})\,]_{kl} := \begin{cases} 1 & \text{if} \quad \text{node } k \text{ is the head of edge } l \\ -1 & \text{if} \quad \text{node } k \text{ is the tail of edge } l \\ 0 & \text{otherwise,} \end{cases}$$

where k and l are the indices running over the node and edge sets, respectively.

A matrix that plays a central role in many graph-theoretic treatments of multi-agent systems is the graph *Laplacian*, defined by

$$\mathcal{L}(\mathcal{G}) := \mathcal{B}(\mathcal{G})\,\mathcal{B}(\mathcal{G})^T; \tag{1}$$

thus the graph Laplacian is a (symmetric) positive semi-definite matrix. Let d_i be the degree of node i and let $\mathcal{D}(\mathcal{G}) := \textbf{Diag}([d_i]_{i=1}^n)$ be the corresponding diagonal degree matrix. It is easy to verify that $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ [11]. As the Laplacian is positive semi-definite, its spectrum can be ordered as

$$0 = \lambda_1(\mathcal{L}(\mathcal{G})) \leq \lambda_2(\mathcal{L}(\mathcal{G})) \leq \dots \leq \lambda_n(\mathcal{L}(\mathcal{G})),$$

with $\lambda_i(\mathcal{L}(\mathcal{G}))$ being the i-th ordered eigenvalue of $\mathcal{L}(\mathcal{G})$. It turns out that the multiplicity of the zero eigenvalue of the graph Laplacian is equal to the number of connected components of the graph [14]. In fact the second smallest eigenvalue $\lambda_2(\mathcal{L}(\mathcal{G}))$ provides a judicious measure of the connectivity of $\mathcal{G}$. For more on the related matrix-theoretic and algebraic approaches to graph theory we refer the reader to [5], [14].

B. Agreement Dynamics

Given a multi-agent system with n agents, we can model the network by a graph $\mathcal{G}$ where nodes represent agents and edges are inter-agent information exchange links.[1] Let $x_i(t) \in \mathbb{R}^d$ denote the state of node i at time t, whose dynamics is described by the single integrator

$$\dot{x}_i(t) = u_i(t), \;\; i = 1, \ldots, n,$$

with $u_i(t)$ being node i's control input. Next, we allow agent i to have access to the relative state information with respect to its neighbors and use it to compute its control. Hence, inter-agent coupling is realized through $u_i(t)$. For example, one can let

$$u_i(t) = -\sum_{i\sim j}(x_i(t) - x_j(t)). \tag{2}$$

The localized rule in (2) happens to lead to the solution of the rendezvous problem, which has attracted considerable attention in the literature [17], [8], [22]. Some other important networked system problems, e.g., formation control [13], [3], [10], consensus or agreement [25], [29], [30], and flocking [35], [32], share the same distributive flavor as the rendezvous problem.

The single integrator dynamics in conjunction with (2) can be represented as the Laplacian dynamics of the form

$$\dot{x}(t) = -\mathbb{L}(\mathcal{G})x(t), \tag{3}$$

where $x(t) = [x(t)_1^T, \; x(t)_2^T, \ldots, \; x(t)_n^T]^T$ denotes the aggregated state vector of the multi-agent system, $\mathbb{L}(\mathcal{G}) := \mathcal{L}(\mathcal{G}) \otimes I_d$, with I_d denoting the d-dimensional identity matrix, and $\otimes$ is the matrix Kronecker product [16]. In fact, if the dynamics of the agent's state is decoupled along each dimension, the behavior of the multi-agent system can be investigated one dimension at a

[1] Throughout this paper we assume that the network is static. As such, the movements of the agents will not cause edges to appear or disappear in the network.

time. Although our results can directly be extended to the case of (3), in what follows, we will focus on the system

$$\dot{x}(t) = -\mathcal{L}(\mathcal{G})x(t), \tag{4}$$

capturing the multi-agent dynamics with individual agent states evolving in $\mathbb{R}$.

III. Observability over the Agreement Protocol

We now endow an observation capability to a subset of agents in the Laplacian dynamics (4); the other agents in the network, the observed nodes, continue to abide by the agreement protocol. In this paper, we use subscripts o and $\bar{o}$ to denote affiliations with observation nodes and nodes that abide by their natural dynamics induced by the agreement protocol, respectively. For convenience, we refer to the nodes that do not serve as observation posts as *observed nodes*; thus the network is partitioned to observers and observed. For example, a *graph* $\mathcal{G}_p$ is the subgraph induced by the observed nodes set $\mathcal{V}_p \subset \mathcal{V}_{\mathcal{G}}$. Observatability designations induce a partition of incidence matrix $\mathcal{B}(\mathcal{G})$ as

$$\mathcal{B}(\mathcal{G}) = \begin{bmatrix} \mathcal{B}_p(\mathcal{G}) \\ \mathcal{B}_o(\mathcal{G}) \end{bmatrix}, \tag{5}$$

where $\mathcal{B}_p(\mathcal{G}) \in \mathbb{R}^{n_p \times m}$, and $\mathcal{B}_o(\mathcal{G}) \in \mathbb{R}^{n_o \times m}$. Here n_p and n_o are the cardinalities of the observed and observer nodes, respectively, and m is the number of edges. The underlying assumption of this partition, without loss of generality, is that observers are indexed last in the original graph $\mathcal{G}$. As a result of (1) and (5), the graph Laplacian $\mathcal{L}(\mathcal{G})$ is given by

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}_p(\mathcal{G}) & l_{po}(\mathcal{G}) \\ l_{po}(\mathcal{G})^T & \mathcal{L}_o(\mathcal{G}) \end{bmatrix}, \tag{6}$$

where

$$\mathcal{L}_p(\mathcal{G}) = \mathcal{B}_p\mathcal{B}_p^T, \ \mathcal{L}_o(\mathcal{G}) = \mathcal{B}_o\mathcal{B}_o^T, \ \text{ and } \ l_{po}(\mathcal{G}) = \mathcal{B}_p\mathcal{B}_o^T.$$

Here we omitted the dependency of $\mathcal{B}, \mathcal{B}_p$, and $\mathcal{B}_o$ on $\mathcal{G}$, which we will continue to do whenever this dependency is clear from the context. As an example, Figure 1 shows an observed network with $\mathcal{V}_o = \{5, 6\}$ and $\mathcal{V}_p = \{1, 2, 3, 4\}$. This gives

$$\mathcal{B}_p = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B}_o = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

and

$$\mathcal{L}_p(\mathcal{G}) = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}, \qquad l_{po}(\mathcal{G}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

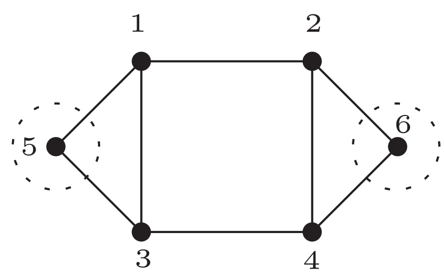


Fig. 1. An observed network with: $\mathcal{V}_p = \{1, 2, 3, 4\}$ and $\mathcal{V}_o = \{5, 6\}$.

The networked system we now consider is the observed agreement dynamics, where the observed nodes evolve through the Laplacian-based dynamics

$$\dot{x}_p(t) = -\mathcal{L}_p \, x_p(t) \tag{7}$$

and the observer node, observe the rest of the network via the linear relationship

$$y(t) = l_{po} x_p(t). \tag{8}$$

Definition 3.1: Let the node "o" be an observer node in $\mathcal{G}$, i.e., $o \in \mathcal{V}_o(\mathcal{G})$. The indicator vector with respect to node o,

$$\delta_o : \mathcal{V}_p \rightarrow \{0, 1\}^{n_p},$$

is such that

$$\delta_o(i) := \begin{cases} 1 & \text{if} \quad i \sim o \\ 0 & \text{otherwise.} \end{cases}$$

We note that each column of l_{po} is an indicator vector, i.e., $l_{po} = [-\delta_{n_p+1}, \dots, -\delta_n]$.

Let d_{io}, with $o \in \mathcal{V}_o$, denote the number of observed nodes adjacent to the observer node o, and define the observed-observer degree matrix

$$\mathcal{D}_{po}(\mathcal{G}) := \mathbf{Diag}([d_{io}]_{i=1}^{n_p}), \tag{9}$$

which leads to the relationship

$$\mathcal{L}_p(\mathcal{G}) = \mathcal{L}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G}), \tag{10}$$

where $\mathcal{L}(\mathcal{G}_p)$ is the Laplacian matrix of the observed graph $\mathcal{G}_p$.

Remark 3.2: We should emphasize the difference between $\mathcal{L}_p(\mathcal{G})$ and $\mathcal{L}(\mathcal{G}_p)$. The matrix $\mathcal{L}_p(\mathcal{G})$ is the principle diagonal sub-matrix of the original Laplacian matrix $\mathcal{L}(\mathcal{G})$ related to the observed nodes, while $\mathcal{L}(\mathcal{G}_p)$ is the Laplacian matrix of the subgraph $\mathcal{G}_p$ induced by the observed nodes. For simplicity, we will write $\mathcal{L}_p$ and l_{po} to represent $\mathcal{L}_p(\mathcal{G})$ and $l_{po}(\mathcal{G})$, respectively, when their dependency on $\mathcal{G}$ is clear from the context.

Since the row sum of the Laplacian matrix is zero, the sum of the i-th row of $\mathcal{L}_p(\mathcal{G})$ and that of $l_{po}(\mathcal{G})$ are both equal to d_{io}, i.e.,

$$\mathcal{L}_p(\mathcal{G})\,\mathbf{1}_{n_p} = \mathcal{D}_{po}(\mathcal{G})\,\mathbf{1}_{n_p} = -l_{po}(\mathcal{G})\,\mathbf{1}_{n_o}, \tag{11}$$

where **1** is a vector with ones at each component.

If there is only one observer node in the network, according to the indexing convention, $\mathcal{V}_o = \{n\}$. In this case, we have $l_{po}(\mathcal{G}) = -\delta_n$ and $\mathcal{D}_{po}(\mathcal{G}) = \mathbf{Diag}(\delta_n)$. For instance, the

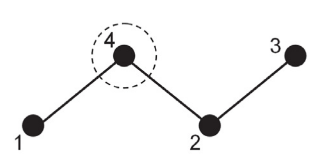


Fig. 2. Path graph with node “4” being the observer.

indicator vector for the node set $\mathcal{V}_p = \{1, 2, 3\}$ in the graph shown in Fig. 2 with respect to the observer $\{4\}$ is $\delta_4 = [\,1,\ 1,\ 0\,]^T$.

Proposition 3.3: If a single node is chosen to be the observer, the original Laplacian $\mathcal{L}(\mathcal{G})$ is related to the Laplacian of the observed graph $\mathcal{L}(\mathcal{G}_p)$ via

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G}) & -\delta_n \\ -\delta_n^T & d_n \end{bmatrix}, \tag{12}$$

where d_n denotes the degree of agent n.

Another way to construct the system matrices $\mathcal{L}_p(\mathcal{G})$ and $l_{po}(\mathcal{G})$ is from the Laplacian of the original graph via

$$\mathcal{L}_p = P_p^T\, \mathcal{L}(\mathcal{G})\, P_p \ \text{ and } \ l_{po} = P_p^T\, \mathcal{L}(\mathcal{G})\, T_{po}, \tag{13}$$

where $P_p \in \mathbb{R}^{n \times n_p}$ is constructed by eliminating the columns of the $n \times n$ identity matrix that correspond to the observers, and $T_{po} \in \mathbb{R}^{n \times n_o}$ is formed by grouping these eliminated columns in a new matrix. For example in Figure 1, these matrices assume the form

$$P_p = \begin{bmatrix} I_4 \\ \mathbf{0}_{2\times 4} \end{bmatrix} \quad \text{and} \quad T_{po} = \begin{bmatrix} \mathbf{0}_{4\times 2} \\ I_2 \end{bmatrix}.$$

Proposition 3.4: If a single node is chosen to be the observer, one has

$$T_{po} = (I_n - \tilde{P})\mathbf{1}_n \quad \text{and} \quad l_{po} = -\mathcal{L}_p \mathbf{1}_{n_p}$$

in (13), where $\tilde{P} = [P_p \ \ \mathbf{0}_{n\times n_o}]$ is the $n \times n$ square matrix obtained by expanding P_p with zero block of proper dimensions.

Proof: The first equality directly follows from the definition of P_p and T_{po}. With out loss of generality assume that the last node is the observer, then $[\,P_p \ \ T_{po}\,] = I_n$. Multiplying both sides by $\mathbf{1}_n$ and noting that $\tilde{P}\,\mathbf{1}_n = P_p \mathbf{1}_{n_p}$, one has $T_{po} = (I_n - \tilde{P})\mathbf{1}_n$.

Moreover,

$$\begin{aligned} l_{po} &= P_p^T \mathcal{L}(\mathcal{G})\{(I - \tilde{P})\mathbf{1}_n\} \\ &= P_p^T \mathcal{L}(\mathcal{G})\mathbf{1}_n - P_p^T \mathcal{L}(\mathcal{G}) P_p \mathbf{1}_{n_p}. \end{aligned}$$

The first term on the right-hand side of the equality is zero as $\mathbf{1}$ belongs to the null space of $\mathcal{L}(\mathcal{G})$; the second term, on the other hand, is simply $\mathcal{L}_p\mathbf{1}$. ∎

IV. Observability Analysis of Single-observer Networks

In this part, we investigate the observability properties of single-observer networks. Following our previously mentioned indexing convention, the index of the observer is assumed to be n. For notational convenience we will subsequently identify matrices A and C with $-\mathcal{L}_p$ and l_{po}^T, respectively. Thus, the system (7) is specified by

$$\dot{x}_p(t) = Ax_p(t) \quad \text{and} \quad y(t) = Cx_p(t). \tag{14}$$

The observability of the observed agreement (14) can be investigated using PHB test [19]. Specifically, (14) is unobservable if and only if there exists an eigenvector ν of A, i.e., $A\nu = \lambda\nu$ for some λ, such that

$$C\nu = 0.$$

Hence, the necessary and sufficient condition for observability of (14) is that none of the eigenvectors of A should be simultaneously orthogonal to C. Additionally, in order to investigate the observability of (14), one can form the observability matrix

$$\mathcal{O} = [\, C^T \;\; (CA)^T \;\; \cdots \;\; (CA^{n_p-1})^T \,]^T. \tag{15}$$

As A is symmetric it can be written in the form $U\Lambda U^T$, where Λ is the diagonal matrix of eigenvalues of A; U on the other hand, is the unitary matrix comprised of A's pair-wise orthogonal unit eigenvectors. Since $C = CUU^T$, by factoring the matrix U from the left in (15), the observability matrix assumes the form

$$\mathcal{O} = \begin{bmatrix} CU \\ CU\Lambda \\ \vdots \\ CU\Lambda^{n_p-1} \end{bmatrix} U^T. \tag{16}$$

In this case, U^T is full rank and its presence does not alter the rank of the matrix product in (16). If one of the columns of U is perpendicular to all the columns of C, then $\mathcal{O}$ will have a row equal to zero and hence rank deficient [36]. On the other hand, in the case of one observer, if any two eigenvalues of A are equal, then $\mathcal{O}$ will have two linear dependent columns, and again, the observability matrix becomes rank deficient. Assume ν_1 and ν_2 are two eigenvectors that correspond to the same eigenvalue and none of them is orthogonal to C. Then $\nu = \nu_1 + c\nu_2$ is also an eigenvector of A for that eigenvalue. This will then allow us to choose $c = -C\nu_1/C\nu_2$, that renders $C\nu = 0$. In other words, we are able to find an eigenvector that is orthogonal to C. Hence, we arrive at the following observation.

Proposition 4.1: Consider an observed network whose evolution is described by (14). This system is observable if and only if none of the eigenvectors of A is (simultaneously) orthogonal to C. Moreover, if A does not have distinct eigenvalues, then (14) is not observable.

Proposition 4.1 is also valid for the case with more than one observer and implies that in any finite time interval, the state of the observed nodes can be monitored by the observers based on local interactions with their neighbors.

Corollary 4.2: The networked system (14) with a single observer is observable if and only if none of the eigenvectors of A is orthogonal to $\mathbf{1}$.

Proof: As shown in Proposition 3.4, the elements of C correspond to column-sums of A, i.e., $C = -\mathbf{1}^T A$. Thus, $C\nu = -\mathbf{1}^T A\nu = -\lambda\,(\nu^T\,\mathbf{1})$. It can be shown that $\lambda \neq 0$. Thereby, $C\nu = 0$ if and only if $\mathbf{1}^T\nu = 0$. ■

Proposition 4.3: If the networked system (14) is unobservable, there exists an eigenvector ν of A such that $\sum_{i\sim n}\nu(i) = 0$.

Proof: Using Corollary 4.2, when the system is unobservable, there exists an eigenvector ν orthogonal to $\mathbf{1}$. As

$$A\,\nu \;=\; \lambda\,\nu,$$

taking the inner product of both sides with $\mathbf{1}$, we obtain

$$\mathbf{1}^T\,(\,A\,\nu\,) \;=\; 0.$$

Using Proposition 3.3 one obtains

$$\nu^T\,\{\,\mathcal{L}(\mathcal{G}_{po}) + \mathcal{D}_{po}(\mathcal{G})\,\}\;\mathbf{1} = \;0.$$

But $\mathcal{L}(\mathcal{G}_{po})\mathbf{1} = 0$ and thereby

$$\nu^T\,\mathcal{D}_{po}(\mathcal{G})\,\mathbf{1} = \nu^T\,\delta_n = 0,$$

or

$$\sum_{i\sim n}\nu(i) = 0.$$

∎

Proposition 4.4: Suppose that the observed system (14) is unobservable. Then one of the eigenvectors of $\mathcal{L}(\mathcal{G})$ has a zero component on the index that corresponds to the observer node.

Proof: Let ν be an eigenvector of A that is orthogonal to **1** (by Corollary 4.2 such an eigenvector exists). Attach a zero to ν; using the partitioning (12), we then have

$$\begin{aligned}\mathcal{L}(\mathcal{G})\begin{bmatrix}\nu\\0\end{bmatrix} &= \begin{bmatrix}A & -\delta_n\\ -\delta_n^T & d_n\end{bmatrix}\begin{bmatrix}\nu\\0\end{bmatrix}\\ &= \begin{bmatrix}\lambda\nu\\ -\delta_n^T\nu\end{bmatrix},\end{aligned}$$

where δ_n is the indicator vector of the observer's neighbors. From Proposition 4.3 we know that $\delta_n^T\nu = 0$. Thus

$$\mathcal{L}(\mathcal{G})\begin{bmatrix}\nu\\0\end{bmatrix} = \lambda\begin{bmatrix}\nu\\0\end{bmatrix}.$$

In the other words, $\mathcal{L}(\mathcal{G})$ has an eigenvector with a zero on the index that corresponds to the observer. ∎

A direct consequence of Proposition 4.4 is the following.

Corollary 4.5: Suppose that none of the eigenvectors of $\mathcal{L}(\mathcal{G})$ has a zero component. Then the observed system (14) is observable for any choice of the observer.

A. Observability and Graph Symmetry

The observability of the interconnected system not only depends on the geometry of the inter-unit information exchange but also on the position of the observer with respect to the graph topology. In this section, we examine the observability of the system in terms of graph-theoretic properties of the network. In particular, we will show that there is intricate relation between observability of (14) and the symmetry structure of the graph as captured by its automorphism group. We first need to introduce a few useful constructs.

Definition 4.6: A permutation matrix is a $\{0,1\}$-matrix with a single nonzero element in each row and column.

Definition 4.7: The observed system (14) is observer symmetric with respect to observer a, if there exists a non-identity permutation J such that

$$JA = AJ, \tag{17}$$

where $A = -\mathcal{L}_p = -P_p^T\,\mathcal{L}(\mathcal{G})\,P_p$ is constructed as in (13). We call the system asymmetric if it does not admit such permutation for any anchor.

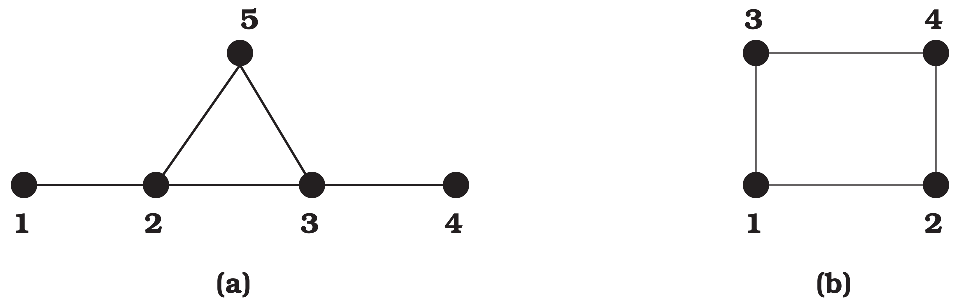


Fig. 3. Interconnected topologies that are observer. symmetric: (a) only with respect to node $\{5\}$, (b) with respect to a observer at any node.

As an example, the graph represented in Fig. 3(a) is observer symmetric with respect to $\{5\}$ but asymmetric with respect to any other observer node set. On the other hand, the graph of Fig. 3(b) is observer symmetric with respect to a single observer located at every node. The utility of the notion of observer symmetry is now established through its relevance to the system theoretic concept of observability.

Proposition 4.8: The observed system (14) is unobservable if it is observer symmetric.

Proof: If the system is observer symmetric then there is a non-identity permutation J such that

$$JA = AJ. \tag{18}$$

Recall that by Proposition 4.1 if the eigenvalues of A are not distinct then (14) is not observable. We thus consider the case where all eigenvalues λ are distinct and satisfy $A\nu = \lambda\nu$; thereby for all eigenvalue/eigenvector pair (λ, ν) one has

$$JA\,\nu = J(\lambda\nu).$$

Using (18) however,

$$A\,(J\nu) = \lambda\,(J\nu)$$

and $J\nu$ is also an eigenvector of A corresponding to the eigenvalue λ. Given that λ is distinct and A admits a set of orthonormal eigenvectors, we conclude that for one such eigenvector ν, $\nu - J\nu$ is also an eigenvector of A. Moreover, $J\,C = J^T C = C$, as the elements of C correspond to the column-sums of the matrix A, i.e., $C = -\mathbf{1}^T A$. Thereby,

$$C(\nu - J\nu) = C\nu - J^T\,C\nu = C\nu - C\nu = 0. \tag{19}$$

This, on the other hand, translates to having one of the eigenvectors of A, namely $\nu - J\nu$, to be orthogonal to C. Proposition 4.1 now implies that the system (14) is unobservable. ■

Proposition 4.8 states that observer symmetry is a sufficient condition for unobservability of the networked system (14). It is instructive to examine whether observer asymmetry leads to an observable system.

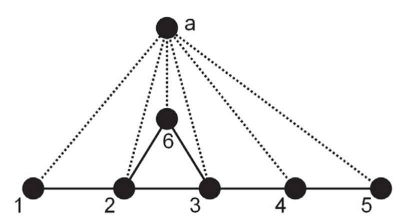


Fig. 4. Asymmetric information topology with respect to the observer $\{a\}$. The subgraph shown by solid lines is the smallest asymmetric graph.

Proposition 4.9: Observer symmetry is not a necessary condition for the system unobservability.

Proof: In Fig. 4, the subgraph shown by solid lines, $\mathcal{G}_p$, is the smallest asymmetric graph [21], in the sense that, it does not admit any non-identity automorphism. Let us augment this graph with the node "a" and connect it to all vertices of $\mathcal{G}_p$. Constructing the corresponding system matrix A (i.e., setting it equal to $-\mathcal{L}_p(\mathcal{G})$), we have

$$-A = \mathcal{L}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G}) = \mathcal{L}(\mathcal{G}_p) + I,$$

where I is the identity matrix of proper dimensions. Consequently, A has the same set of eigenvectors as $\mathcal{L}(\mathcal{G}_p)$. Since $\mathcal{L}(\mathcal{G}_p)$ has an eigenvector orthogonal to $\mathbf{1}$, A also has an eigenvector that is orthogonal to $\mathbf{1}$. Hence, the observed system is not observable. Yet, the system is not symmetric with respect to a; more on this in Section IV-B. ■

It is intuitive that a highly connected observer will result in faster convergence for the observer to the state of the observed nodes. However, a highly connected observer also increases the chances that a symmetric graph, with respect to observer, emerges. A limiting case for this latter scenario is the complete graph. In such a graph, $n-1$ observers are needed to make the corresponding dynamic system, observable. This requirement is of course not generally desirable as it means that the observer group include all nodes except for one node! The complete graph is “the worse” case configuration as far its single-node observability properties. Generally at most $n-1$ observers are needed to make any information exchange network observable. In the meantime, a path graph with a observer at one end is observable. Thus it is possible to make a complete graph observable by keeping the links on the longest path between a observer and all other nodes, deleting the unnecessary information exchange links to break its inherent symmetry. This procedure is not always feasible; for example a star graph is not amenable to such graphical alterations.

B. Observer Symmetry and Graph Automorphism

In Section IV-A we discussed the relationship between observer symmetry and observability. In this section we will further explore the notion of observer symmetry with respect to graph automorphisms.

Definition 4.10: An automorphism of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a permutation ψ of its node set such that

$$(\psi(i)\,, \psi(j)) \in \mathcal{E}_{\mathcal{G}} \quad \Longleftrightarrow \quad (i, j) \in \mathcal{E}_{\mathcal{G}}.$$

The set of all automorphisms of $\mathcal{G}$, equipped with the composition operator, constitutes the automorphism group of $\mathcal{G}$; note that this is a “finite” group. It is clear that the degree of a node remains unchanged under the action of the automorphism group, i.e., if ψ is an automorphism of $\mathcal{G}$ then $d_v = d_{\psi(v)}$ for all $v \in \mathcal{V}_{\mathcal{G}}$.

Proposition 4.11 ([5]): Let $\mathcal{A}(\mathcal{G})$ be the adjacency matrix of the graph $\mathcal{G}$ and ψ a permutation on its node set $\mathcal{V}$. Associate with this permutation, the permutation matrix Ψ such that

$$\Psi_{ij} := \begin{cases} 1 & \text{if} \quad \psi(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ is an automorphism of $\mathcal{G}$ if and only if

$$\Psi\, \mathcal{A}(\mathcal{G}) \,=\, \mathcal{A}(\mathcal{G})\, \Psi.$$

In this case, the least positive integer z for which $\Psi^z = I$ is called the order of the automorphism. Recall that from Definition 4.7 observer symmetry for (14) corresponds to having

$$JA = AJ,$$

where J is a non-identity permutation.From Proposition 3.3 however,

$$A = -(\mathcal{L}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G})).$$

Thus using the identity $\mathcal{L}(\mathcal{G}_p) = \mathcal{D}(\mathcal{G}_p) - \mathcal{A}(\mathcal{G}_p)$ one has

$$\begin{aligned} & J\ \{\mathcal{D}(\mathcal{G}_p) - \mathcal{A}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G})\} \\ & = \{\mathcal{D}(\mathcal{G}_p) - \mathcal{A}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G})\}\ J. \end{aligned} \tag{20}$$

Pre and post multiplication of (a permutation matrix) J, does not change the structure of diagonal matrices. Also, all diagonal elements of $\mathcal{A}(\mathcal{G})$ are zero. We can thereby rewrite (20) as two separate conditions,

$$J\mathcal{D}_p(\mathcal{G}) = \mathcal{D}_p(\mathcal{G})\, J \quad \text{and} \quad J\mathcal{A}(\mathcal{G}_p) = \mathcal{A}(\mathcal{G}_p)J, \tag{21}$$

with $\mathcal{D}_p(\mathcal{G}) := \mathcal{D}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G})$. The second equality in (21) states that sought after J in (17) is in fact an automorphism of $\mathcal{G}_p$.

Proposition 4.12: Let Ψ be the permutation matrix associated with ψ. Then $\Psi\, \mathcal{D}_p(\mathcal{G}) = \mathcal{D}_p(\mathcal{G})\, \Psi$ if and only if

$$d_i + \delta_n(i) = d_{\psi(i)} + \delta_n(\psi(i)).$$

In the case where ψ is an automorphism of $\mathcal{G}_p$, this condition simplifies to

$$\delta_n(i) = \delta_n(\psi(i)).$$

Proof: Using the properties of permutation matrices, one has

$$[\Psi\mathcal{D}_p(\mathcal{G})]_{ik} = \sum_t \Psi_{it}\mathcal{D}_{tk} = \begin{cases} d_k + \delta_n(k) & \text{if} \quad i \to k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$[\mathcal{D}_p(\mathcal{G})\Psi]_{ik} = \sum_t \mathcal{D}_{it}\,\Psi_{tk} = \begin{cases} d_i + \delta_n(i) & \text{if} \quad i \to k \\ 0 & \text{otherwise.} \end{cases}$$

For these matrices to be equal element-wise, one needs to have $d_i + \delta_n(i) = d_k + \delta_n(k)$ when $\psi(i) = k$. The second statement in the proposition follows from the fact that the degree of a node remains invariant under the action of the automorphism group. ■

The next two results follow immediately from the above discussion.

Proposition 4.13: The interconnected system (14) is observer symmetric if and only if there is a non-identity automorphism for $\mathcal{G}_p$ such that the indicator function remains invariant under its action.

Corollary 4.14: The interconnected system (14) is observer asymmetric if the automorphism group of the observed subgraph only contains the trivial (identity) permutation.

C. Observability of special graphs

In this section we investigate the observability of ring and path graphs.

Proposition 4.15: A ring graph, with only one observer, is never observable.

Proof: With only one observer in the ring graph, the observed graph $\mathcal{G}_p$ becomes the path-graph with one non-trivial automorphism, i.e., its mirror image. Without loss of generality, choose the first node as the observer and index the remaining observed nodes by a clock-wise traversing of the ring. Then the permutation $i \rightarrow n - i + 2$ for $i = 2, \ldots n$, is an automorphism of $\mathcal{G}_p$. In the meantime, the observer "1" is connected to both node 2 and node n; hence $\delta_n = [\,1,\, 0,\, \cdots,\, 0,\, 1\,]^T$ remains invariant under the permutation. Using Proposition 4.13, we conclude that the corresponding system (14) is observer symmetric and thus unobservable. ■

Proposition 4.16: A path graph is observable for any choice of observer if and only if it is of even order.

Proof: Suppose that the path graph is of odd order; then choose the middle node "$\frac{n+1}{2}$", as observer. Note that $\psi(k) = n - k + 1$ is an automorphism for the floating subgraph. Moreover, the observer is connected to nodes $\frac{n+1}{2} - 1$ and $\frac{n+1}{2} + 1$, and $\psi(\frac{n+1}{2} - 1) = \frac{n+1}{2} + 1$. Thus

$$\delta_n = [\,0,\, \cdots,\, 0,\, 1,\, 1,\, 0,\, \cdots\, 0\,]^T$$

remains invariant under the permutation ψ and the system is unobservable. The converse statement follows analogously. ■

Hence although in general observer symmetry is sufficient- yet not necessary- condition for unobservability of (14), it is necessary and sufficient for unobservability of the path graph.

Corollary 4.17: A path graph with a single observer is observable if and only if it is observer asymmetric.

V. FORMATION ESTIMATION VIA LOCAL INFORMATION-EXCHANGE

In this section, we consider a scenario where a group of spacecraft reach a desired formation via a neighboring information-exchange mechanism. During this process, one of the nodes in the network serves as the observation node and has access to local measurement of its neighboring spacecraft state(s).

Consider the agreement protocol with the Laplacian of the information exchange network partitioned in the form defined in (12). It is assumed that a Laplacian-based formation control algorithm is running on the spacecraft fleet with change of variable from x to $x - x_r$, where x_r specifies the desired formation configuration. We can then write the equation of motion of the formation in the form of an LTI system as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dz_r \\ y(t) &= Cx(t) \end{aligned} \tag{22}$$

where

$$A = -\mathcal{L}(\mathcal{G}_p) + \mathcal{D}_{po}(\mathcal{G}), \quad B = \delta_n, \quad D = [\mathcal{B}_p \ \ \delta_n],$$

and $C = \delta_n^T$ denotes the local observation geometry accessible to the observer node.

Here we assume that the formation is not controlled by exogenous signals from the observation post, that is, $u = 0$. The term z_r refers to the desired relative formation and is calculated from desired final configuration via

$$z_r = \begin{bmatrix} \mathcal{B}_p & \delta_f \\ 0 & -1 \end{bmatrix} x_r.$$

The discretized model of (22) in presence of measurement and process noise, assumes the form

$$\begin{aligned} x(k+1) &= \bar{A}x(k) + \bar{D}z_r + v(k), \\ y(k) &= \bar{C}x(k) + w(k), \end{aligned} \tag{23}$$

where v and w are zero mean Gaussian processes with covariances Q and R, respectively.

In the proposed setup, the observer spacecraft runs a discrete Kalman filter to estimate the entire state of the spacecraft formation. Between the communication or measurement steps, the estimate of the formation state and its covariance evolve according to

$$\begin{aligned} x(k+1^-) &= \bar{A}\,x(k^+) + \bar{D}z_r \\ P(k+1^-) &= \bar{A}\,P(k^+)\,\bar{A}^T + Q. \end{aligned} \tag{24}$$

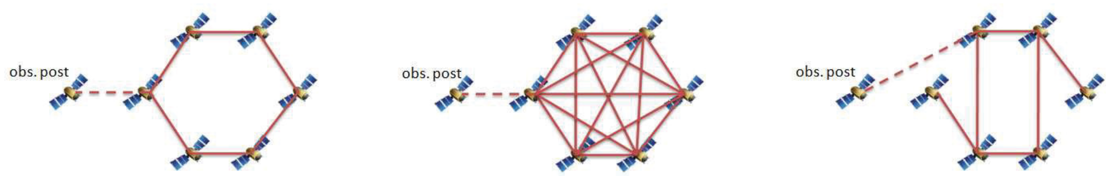
At each communication step, the Kalman gain is computed and state estimate and covariance matrix are updates according to

$$\begin{aligned} K(k) &= P(k^-)\bar{C}^T(\bar{C}\,P(k^-)\,\bar{C}^T + R)^{-1} \\ \hat{x}(k^+) &= \hat{x}(k^-) + K(k)(y(k) - \bar{C}\hat{x}(k^-)) \\ P(k^+) &= P(k^-) - K\,\bar{C}\,P(k^-). \end{aligned} \tag{25}$$

Here we should emphasis that *the observability of the system from the observer node guarantees the convergence of the Kalman filter as it operates on a local information-exchange mechanism induced by the observed formation.* We also like to point out that, rather counter-intuitively, more information-exchange capability for the formation does not always translate to a *more observable* network. Figure (5) depicts a few examples of unobservable networks on a distributed space system consisting of seven spacecraft.

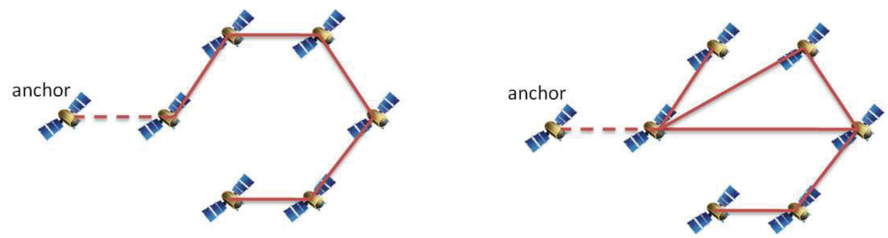
We have simulated the proposed formation estimation algorithm on a group of seven spacecraft moving to a desired formation. In each scenario, we used a different observation post and estimated the state of the whole system viewed from the observer spacecraft, accessing only local measurement of its neighboring nodes.

Figure (6) shows two of the formations considered in our simulations. Figure (7) depicts the state estimation error for these two networks as the percentage of the formation size in the



(a) (b) (c)

Fig. 5. Unobservable formations.



(a) (b)

Fig. 6. Observable formations.

presence of, respectively, 5% and 10% Gaussian measurement and process noise variances. As one can see from this example, the estimation error decreases rapidly and reaches the corresponding measurement error bound. We also note that the path network is slower to reach the desired error bound, exhibiting an inherit delay in information fusion associated with its large diameter.

VI. Conclusions

In this paper, we considered the observability of a distributed space system from a single observation node. Such global observability property has been facilitated via a local protocol induced by the agreement protocol. We first derived a set of transformations that could be employed to derive the system matrices for scenarios where one or more of the nodes serve as observation nodes. The other nodes in the graph (the observed nodes) are assumed to update their states according to their relative states with their neighbors. In such a setting, we studied the observability of the resulting dynamic system. We then showed that there is a intricate relationship between the unobservability of the corresponding multi-agent system and various network-theoretic properties of the network. In particular, we pointed out the importance of the network automorphism group in the observability properties of distributed and interconnected systems. The ramifications of this correspondence were then explored in the context of distributed estimation for multiple spacecraft systems that operate on a local-information exchange mechanism. The results of the present work point to a promising research direction at the intersection of space systems, networks, and estimation theory that aims to study estimation and control issues for multiple spacecraft systems from a network-theoretic outlook.

Acknowledgments

This research was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration and funded through the Director's Research and Development Fund Program. The first two authors thank Magnus Egerstedt for helpful discussions on the topics related to the present paper.

References

[1] T. Balch and R. C. Arkin. Behavior-based formation control for multi-robot teams, *IEEE Transactions on Robotics and Automation*, 14: 926-939, 1998.

[2] B. Bamieh, F. Paganini, and M. Dahleh. Distributed control of spatially-invariant systems, *IEEE Transactions on Automatic Control*, 47 (7): 1091-1107, 2002.

[3] R. W. Beard, J. R. Lawton, and F. Y. Hadaegh. A coordination architecture for spacecraft formation control, *IEEE Transactions on Control Systems Technology*, (9) 6: 777–790, 2001.

[4] D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation*, Prentice Hall, Englewood Cliffs, NJ, 1989.

[5] N. Biggs. *Algebraic Graph Theory*, Cambridge University Press, 1993.

[6] B-D. Chen and S. Lall. Dissipation inequalities for distributed systems on graphs, *IEEE Conference on Decision and Control*, December 2003.

[7] J. Cortés and F. Bullo. Coordination and geometric optimization via distributed dynamical systems, *SIAM Journal of Control and Optimization*, (45) 5: 1543-1574, 2005.

[8] J. Cortés, S. Martínez, and F. Bullo. Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions, *IEEE Transactions on Automatic Control* (51) 8: 1289–1298, 2006.

[9] R. D'Andrea and G. E. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*, (48) 9: 1478-1495, 2003.

[10] J. Desai, J. Ostrowski, and V. Kumar. "Controlling formations of multiple mobile robots," *IEEE International Conference on Robotics and Automation*, May 1998.

[11] R. Diestel. *Graph Theory*, Springer, 2000.

[12] C. Fall, E. Marland, J. Wagner, and J. Tyson (Editors). *Computational Cell Biology*, Springer, 2005.

[13] J. Fax and R. Murray. Information flow and cooperative control of vehicle formations, *IEEE Transactions on Automatic Control* (49): 1465–1476, 2004.

[14] C. Godsil and G. Royle. *Algebraic graph theory*. Springer, 2001.

[15] Y. Hatano and M. Mesbahi. Agreement over random networks. *IEEE Transactions on Automatic Control*, (50) 11: 1867-1872, 2005.

[16] R. A. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[17] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Transactions on Automatic Control* (48) 6: 988–1001, 2003.

[18] M. Ji, A. Muhammad, and M. Egerstedt, "Leader-based multi-agent coordination: controllability and optimal control," *American Control Conference*, June 2006.

[19] T. Kailath. *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.

[20] Y. Kim, M. Mesbahi, and F. Y. Hadaegh. Multiple-spacecraft reconfigurations through collision avoidance, bouncing, and stalemates, *Journal of Optimization Theory and Applications*, (122) 2: 323-343, 2004.

[21] J. Lauri and R. Scapellato. *Topics in Graph Automorphism and Reconstruction*, Cambridge, 2003.

[22] Z. Lin, M. Broucke, and B. Francis. Local control strategies for groups of mobile autonomous agents, *IEEE Transactions on Automatic Control* (49) 4: 622–629, 2004.

[23] D. Luenberger. *Optimization by Vector Space Methods*. New York: Wiley, 1969.

[24] R. Merris. Laplacian matrices of graphs: a survey, *Linear Algebra and its Applications*, 197,198: 143-176, 1994.

[25] M. Mesbahi. "State-dependent graphs," *IEEE Conference on Decision and Control*, December 2003.

[26] M. Mesbahi. State-dependent graphs and their controllability properties, *IEEE Transactions on Automatic Control*, (50) 3: 387 - 392, 2005.

[27] M. Mesbahi and F. Y. Hadaegh. Formation flying control of multiple spacecraft via graphs, matrix inequalities, and switching, *AIAA Journal of Guidance, Control, and Dynamics*, (24) 2: 369-377, 2001.

[28] L. Moreau. Stability of multiagent systems with time-dependent communication links, *IEEE Transactions on Automatic Control*, (50) 2: 169 - 182, 2005.

[29] R. Olfati-Saber and R. M. Murray. "Agreement problems in networks with directed graphs and switching topology," *IEEE Conference on Decision and Control*, December 2003.

[30] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays, *IEEE Transactions on Automatic Control* (49) 9: 1520–1533, 2004.

[31] R. Olfati-Saber and J. S. Shamma. Consensus filters for sensor networks and distributed sensor fusion, *IEEE Conference on Decision and Control*, December 2005.

[32] R. Olfati-Saber. Flocking for multi-agent dynamic systems: algorithms and theory, *IEEE Transactions on Automatic Control* (51) 3: 401–420, 2006.

[33] K. Ogata. *Modern Control Engineering*, Prentice Hall, Englewood Cliffs, NJ, 2002.

[34] A. Rahmani and M. Mesbahi. "On the controlled agreement problem," *American Control Conference*, June 2006.

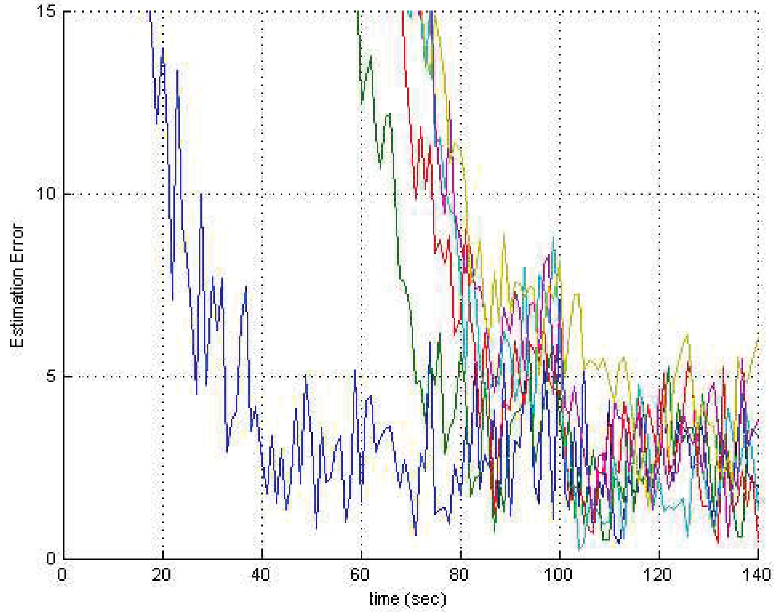
[35] H. Tanner, A. Jadbabaie, and G. Pappas, "Stable flocking of mobile agents, part II : dynamic topology," *IEEE Conference on Decision and Control*, December 2003.

[36] H. G. Tanner, “On the controllability of nearest neighbor interconnections,” *IEEE Conference on Decision and Control*, December 2004.

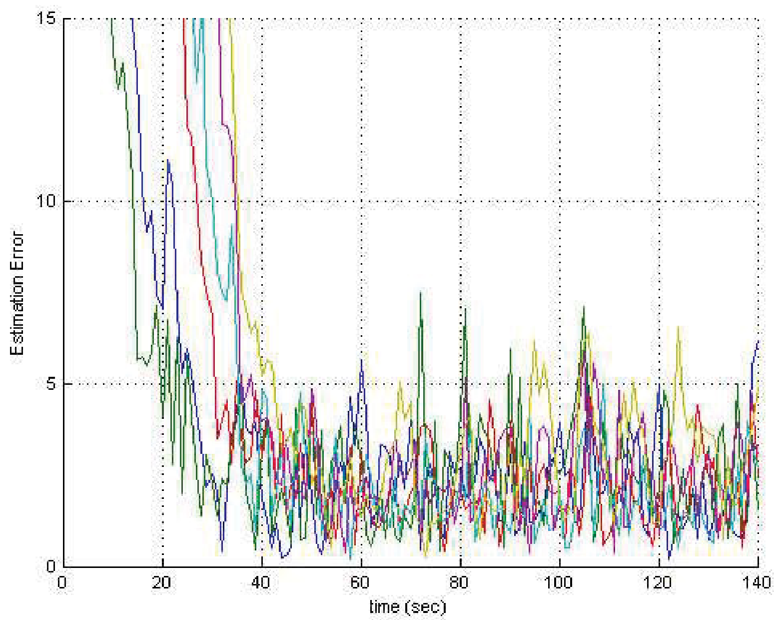
[37] G. Walsh, H. Ye, and L. Bushnell. “Stability analysis of networked control systems,” *American Control Conference*, June 1999.

[38] P. K. C. Wang and F. Y. Hadaegh. Coordination and control of multiple microspacecraft moving in formation, *Journal of the Astronautical Sciences*, 44: 315-355, 1996.

[39] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems and Control Letters*, (53): 65-78, 2004.



(a)



(b)

Fig. 7. State estimation error in terms of the percentage of the formation size: (a) for the network shown in Fig. 6(a), (b) for the network shown in Fig. 6(b).